In this research report, we aim at carrying out a comparative study between two theoretical approaches that allows carrying out cognitive analysis from the subjects’ performance: Theory of Register of Semiotic Representation and the Onto-Semiotic Approach of mathematical cognition and instruction. In order to carry out this study, we analysed the performance of a future high school teacher in a task related to the differentiability of the absolute value function. As a result of this study, the complementarities between these two theoretical perspectives, which might allow more complete and detailed analysis of the students’ performance, are evidenced.

INTRODUCTION

One of the main concerns of the research community on the Mathematics Education is determining which are the difficulties that learners face on their way to understanding, and therefore, learning, of mathematical notions. This interest is reflected in the fact that one of the main focuses of research within our scientific discipline has been the features of the learner’s cognitive activity. Currently, there are several theoretical positions that allow carrying out cognitive analysis (of students, prospective teachers or teachers) depending on what is desired to observe and which is the concerned mathematical notion (Duval, 2006; Asiala, Brown, DeVries, Dubinsky, Mathews & Thomas, 1996; Godino, Batanero & Font, 2007). However, the complex nature of the subjects’ learning phenomena has directed research groups to make efforts to revise and find possible complementarities between theoretical and methodological approaches that allow providing more detailed and precise explanations of such learning processes.

In this research report, we present a comparative study between two theoretical approaches, the Theory of Register of Semiotic Representation (TRSR) and the Onto-Semiotic Approach (OSA) of mathematical cognition and instruction, which allows carrying out cognitive analysis from the subjects’ performance. In order to carry out this study, and following the proposed methodology for the works within the framework of the networking of theories, we analysed the performance of a future high school teacher in a task related to the differentiability of the absolute value function. As a result of this study, the complementarities between these two
Theoretical perspectives, which might allow more complete and detailed analysis of the students’ performance, are evidenced.

**THEORETICAL FRAMEWORKS**

**Theory of Register of Semiotic Representation (TRSR)**

In the context of cognitive psychology, the notion of representation plays an important role regarding the acquisition and the treatment of an individual’s knowledge. As Duval (1995) points out: “There’s no knowledge that can be mobilized by an individual without a representation activity” (p. 15).

The comprehension of the theory on registers of representation requires the consideration of three key characteristics:

1. There are as many different semiotic representations of the same mathematical object, as semiotic registers utilized in mathematics.

2. Each different semiotic representation of the same mathematical object does not explicitly state the same properties of the object being represented. What is being explicitly stated is the content of the representation.

3. The content of semiotic representations must never be confused with the mathematical objects that these represent.

One of the specificities of semiotic representations consists of its dependence on an organized system of signs such as language, numerical writing, symbolic writing and Cartesian graphs. Consequently, all semiotic representations must be considered, primarily, based on the register where it was produced; then, based on what it explicitly does and what it cannot represent; secondly, based on what it explicitly does and what it cannot represent of the properties of the object of knowledge being analyzed; and finally, based on the object itself to which it refers to.

Another of the essential specificities of the semiotic representations is the cognitive operation of conversion of the representations from one system into another, in other words, the transformation of semiotic representations into other semiotic representations. Duval (Ibid, p. 17) points it out as: “The notion of semiotic representation presupposes the consideration of different semiotic systems and a cognitive operation of conversion of the representations form one system into another”. This conversion operation has been considered as a change of form: moving from a verbal statement into an algebraic operation, or draw the curve of a second-degree equation. These examples illustrate the change in the form that knowledge is represented.

It is important to point out that there are two fundamental cognitive activities within the TRSR: treatment and conversion. The activity of treatment consists of a transformation carried out in the same register, in other words, only one register is mobilized. The activity of conversion, on the other hand, consists of the mobilization
from one register into another, where the articulation of representation becomes fundamental. According to Duval (Ibid), the study of the activity of conversion makes it possible to comprehend the close relation between ‘noesis’ and ‘semiosis’, which is essential in intellectual learning. However, it must be taken into account that the operation of conversion brings some difficulties, one regarding the fact that the representation of the source register does not have the same content as the destination register. Another difficulty lies in the treatment, which becomes complex by the use of the register of natural language and those registers that allow ‘visualizing’ (graphs, geometrical shapes, etc.).

Semiotic systems that allow studying the pairs ‘representation, knowledge’, must satisfy the three cognitive activities related to representation: 1) Constituting a trace or an assembling of traces that are identifiable as a representation of an object or thing; 2) Transforming representations according to the rules typical to the system in order to obtain other representations that might provide more knowledge to the initial representations; y 3) Converting representations produced in a system of representation into another system, so that the latter, allow making other meanings explicit to what is being represented. Not all semiotic systems allow these three cognitive activities. Semiotic systems that do allow said cognitive activities are what Duval (Ibid) calls registers of semiotic representation. These registers of semiotic representation constitute the degrees of freedom that a subject has to objectify an idea that is initially confusing, a beating feeling, taking advantage of information, or communicating with an interlocutor.

The Onto-Semiotic Approach of mathematical cognition and instruction

The Onto-Semiotic Approach (OSA) to cognition and mathematical instruction is a theoretical and methodological framework that has been developed since 1994 by Godino and colleagues (Godino, Batanero & Font, 2007; Font, Godino & Gallardo, 2013). Such theoretical framework includes an epistemological model about mathematics, on anthropological and sociocultural bases, a cognitive model on semiotic bases from a pragmatic nature, and an instructional model coherent to the others mentioned above. Then, there are six facets or dimensions that are considered in OSA, for the study of the processes of teaching and learning, in relation to a specific mathematical content (Godino, Batanero & Font, 2007): epistemic, cognitive, affective, interactional, meditational and ecological. The cognitive facet refers to the development of personal meanings (learning of students).

Within the onto-semiotic approach, the notion of ‘system of practices’ plays an important role for the teaching and learning of mathematics. Godino & Batanero (1994) refer to the system of practices as “any performance or manifestation (linguistic or not) carried out by someone in order to solve mathematical problems, to communicate the solution to others, to validate the solution and to generalize it to other contexts and problems” (p. 334). Font, Godino & Gallardo (2013), point out that mathematical practices can be conceptualized as the combination of an operative
practice, through which mathematical texts can be read and produced, and a
discursive practice, which allows the reflection on operative practices. These
practices can be carried out by one person (system of personal practices) or shared
within an institution (system of institutional practices).

Now then, within the OSA, certain pragmatism is adopted since mathematical objects
are considered as entities that emerge from the systems of practices carried out in a
field of problems (Godino & Batanero, 1994). Font, Godino & Gallardo (2013) put it
this way: “Our ontological proposal originates from mathematical practices, and
these become the basic context from which individuals gain experience and
mathematical objects emerge from. Consequently, the object gains a status originated
from the practices that precede it” (p. 104). Ostensive objects (symbols, graphs, etc.)
and non-ostensive objects (concepts, propositions, etc.) intervene in mathematical
practices, which we evoke while doing mathematics and are represented in a textual,
oral, graphic, symbolic and even gestural way. New objects emerge from the systems
of operative and discursive mathematical practices and these show their organization
and structure (Godino, Batanero & Font, 2007). If the systems of practices are shared
within the core of an institution, then the emerging objects will be considered as
‘institutional objects’, while, on the other hand, if such systems correspond to one
person, then these will be considered as ‘personal objects’. The emergence of a
personal object is progressive during the history of a subject, as a consequence of
experience and learning, while the emergence of an institutional object is progressive
over time.

The notion of ‘system of practices’ is useful for a certain type of macro didactic
analysis. For a ‘finer’ analysis of mathematical activity, the following typology of
primary mathematical objects that intervene in the systems of practices, have been
introduced in the OSA: 1) situations-problems (extra-mathematical applications,
exercises,…); 2) linguistic elements (terms, expressions, notations, graphs,…) in
diverse registers (written, oral, gestural,…); 3) concepts/definitions (introduced
through definitions or descriptions: line, point, number, average, function,
derivative,…); 4) propositions/properties (statements about concepts,…); 5)
procedures (algorithms, operations, calculation techniques,…); and 6) arguments
(statements used to validate or explain propositions and procedures, deductive or of
another type,…). When an agent performs and evaluates a mathematical practice, a
conglomerate formed by situation-problems, languages, concepts, propositions,
procedures and arguments, is activated. These primary mathematical objects are
connected with each other, forming intervening networks of objects, emerging from
the systems of practices, which in OSA are known as configurations. These
configurations can be socio-epistemic (networks of institutional objects) or cognitive
(networks of personal objects).
METHODOLOGY

We use the proposed methodology for studies on networking of theories, which suggest for this type of studies, among other things, to select a problem or particular case and analyze this case or problem under the theoretical perspectives involved in the study. In our case, we select a task on the differentiability of the absolute value function and the solution provided by a student of university level to this task. This student, whom we refer to as Juliet, was enrolled in the final modules (eighth semester) of the degree in mathematics teaching offered by the Autonomous University of Yucatan (UADY) in Mexico. So, she had studied differential calculus in the first semester of their degree course, and had subsequently completed other modules related to mathematical analysis (integral calculus, vector calculus, differential equations, etc.). She had also studied subjects related to the teaching of mathematics. Both the task and the solution provided by Juliet can be found in the study of Pino-Fan (2014). Juliet's answer was chosen intentionally due to its cognitive complexity.

The task and the solution provide by Juliet

This task (Figure 1) has been studied in an investigation on teacher training (Pino-Fan, 2014).

| Task | Examine the function $f(x) = |x|$ and its graph. |
|------|--------------------------------------------------|
|      | ![Graph of the absolute value function](image)    |

a) For what values of $x$ is $f(x)$ derivable?

b) If possible, calculate $f'(2)$ and draw a graphic representation of your solution. If not possible, explain why.

c) If possible, calculate $f'(0)$ and draw a graphic representation of your solution. If not possible, explain why.

Figure 1: Task about derivability of the absolute value function

The Juliet’s solution translated from Spanish to English language is presented in Figure 2. The answer of Juliet was analyzed from two perspectives (TRSR and OSA). From the point of view of the TSRS, the analysis focused on the identification and description of the semiotic registers of representation mobilized by Juliet, and the
study of congruence between the activities of treatments or conversions/passages. From the point of view of the OSA, the mathematical practice of Juliet and the cognitive configuration (linguistic elements, concepts/definitions, properties/propositions, procedures and arguments) that mobilized as part of such a practice, were characterized.

| a) In the first instance, the function $f(x) = |x|$ is not differentiable, because the graph has a peak at the point $x = 0$. If it is considered as a function of the type $f(x) = \begin{cases} x & \forall x \geq 0 \\ -x & \forall x < 0 \end{cases}$, the function would be differentiable in the whole domain, in other words, $(-\infty, \infty)$. |
| --- |
| b) As mentioned above, the function is not differentiable, therefore $f'(2)$ cannot be calculated and if it was differentiable, its graphic representation would be a point on the Cartesian plane. Considering this type of function: $f(x) = \begin{cases} x & \forall x \geq 0 \\ -x & \forall x < 0 \end{cases}$, the derivative would be $f'(x) = 1$, and then it would represent a point. |
| Figure 2: Juliet’s solution to the task |
| c) As mentioned above, if we consider the function $f(x) = \begin{cases} x & \forall x \geq 0 \\ -x & \forall x < 0 \end{cases}$, the derivative would be $f'(x) = 1$ and the graphic representation would be. |

**ANALYSIS FROM BOTH PERSPECTIVES**

For reasons of space, we present in this research report a summary of analyses carried out from both approaches. The analysis carried out both with TRSR and OSA, show deficiencies in Juliet’s mathematical activity, related to the lack of connection of the interpretations and treatments that she makes in the graphic and symbolic representations of the absolute value function. Through the lens of the TRSR could be observed that the Juliet’s answers for the items a), b) and c), show that she knows the definition of the absolute value function, and that she can express it in the symbolic register. But regarding the derivative function, she shows deficiencies, because even though she answers that if the graphic of the function presents a corner or peak on $x=0$, then the function is not derivable, in her upcoming arguments some confusions are perceived regarding the domain and little detailed graphic of $f'(2)$. She manages representing for the non-negative values of $x$, the derivative of $f$ in symbols, but does not represent it graphically, which could be explained by insufficient knowledge. Although cognitively, she had the symbolic and graphic registers, she does not succeed in the mathematical knowledge of the derivative function, which might be because it appears in an implicit way in the task. In conclusion, a disconnection between the graphic and symbolic registers in which Juliet stands to give her answers is observed. In this sense, we can conclude that Juliette does not carry out a cognitive operation of coordination and articulation between such registers.
With the lens of OSA we observe that Juliet begins her practice based on a visual justification to answer, although wrongly, to subtask a), pointing out the existence of a ‘peak’ at the point of domain of the function \( x=0 \). From the beginning of her practice, we can observe that Juliet confuses the non-derivability (local) at a point of domain of the absolute value function with her misconception of non-derivability of the function (global). Later, Juliet writes the symbolic definition, by parts, of the absolute value function. We could say that, in a certain way, such definition is correct, however, she does not make crucial considerations, for example, that the point of domain of the function \( x = 0 \) belongs to both \( f(x) = x \) and \( f(x) = -x \). This fact leads her to a cognitive conflict that is shown in her sentence “If it is considered as a function of the type... the function would be differentiable in the whole domain, in other words, \((-\infty, \infty)\)”. This cognitive conflict generated from her visual interpretation of the graph of the function (the function is not derivable since it has a peak in \( x = 0 \)) in contraposition to her interpretation of the symbolic definition, by parts, of the function (she considers that \( f(x) = x \) exclusively for \( x \geq 0 \)), is what leads her to give incorrect answers to the other subtasks.

**CONCLUSION**

The results of the comparison of analysis show that between these two theoretical perspectives there are complementarities that would allow performing more precise and ‘finer’ cognitive analysis, from the subjects’ production. In this way, it is plausible to provide better explanations about the aspects that make it possible or impossible to comprehend mathematical notions. While the analysis from the OSA perspective focused on the subjects’ mathematical practices, and mathematical objects, processes and their meanings, that emerge from such practices, the TRSR focused its analysis primarily on the registers of representation that the subject mobilizes in his/her productions. In this way, the methodology proposed by TRSR can be considered as more ‘global’, in the sense that the subjects’ cognitive activity is analyzed without performing valuations from a mathematical point of view, as it is done with the tools of OSA. So, the OSA provides a level of analysis of the subject’s cognitive activity that shows mathematical objects that are involved in the processes of treatment and conversion/passages between registers of semiotic representation. This level of analysis complements the analysis carried out using the tools of TRSR, because with the tools ‘configuration of objects and processes’ and ‘semiotic function’, the contents of representations become explicit and are utilized as part of such cognitive activity. It is clear that the registers of representation are implicitly involved in semiotic functions; however, these emphasize the mathematical content of the representation. However, should be noted that within the OSA there is not systematization for the analysis of linguistic elements. As a part of the methodology
proposed by OSA, language signs –linguistic elements– can be identified, but these different languages could make reference both to register of semiotic representation and semiotic systems. TRSR makes a clear distinction between register of semiotic representation and semiotic system. Thus, the notion of register of semiotic representation of TRSR, complements and enriches the notion of linguistic elements of OSA, by making a very clear distinction between register and semiotic system, and systematizing the analysis of such registers.

Finally, these complementarities between the TRSR and OSA show us guidelines for creating a methodology to perform cognitive analysis most ‘comprehensive’ and ‘profound’, but that is the next step in our research. We are convinced that the relationship between notions of mathematical objects (as considered in OSA) and semiotic representation (as considered in TRSR), are essential for the analysis and characterization of mathematical knowledge.

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References


